A

Values of Physical Constants[1]

Avogadro's constant,

$$N_{\mathrm{Av}} = 6.02214 \times 10^{23}\ \mathrm{mol}^{-1}\,.$$

Molar ideal gas constant,

$$\begin{aligned} R &= 8.3145\ \mathrm{J\,K^{-1}\,mol^{-1}} = 0.082056\ \mathrm{liter\ atm\ K^{-1}\ mol^{-1}} \\ &= 1.9872\ \mathrm{cal\,K^{-1}\,mol^{-1}}\,. \end{aligned}$$

The magnitude of an electron's charge,

$$e = 1.602177 \times 10^{-19}\,\mathrm{C}\,.$$

Planck's constant,

$$h = 6.62608 \times 10^{-34}\ \mathrm{J\ s}.$$

Boltzmann's constant,

$$k_B = 1.38066 \times 10^{-23}\ \mathrm{J\,K^{-1}}.$$

The rest-mass of an electron,

$$m_e = 9.10939 \times 10^{-31}\ \mathrm{kg}.$$

The rest-mass of a proton,

$$m_p = 1.672623 \times 10^{-27}\ \mathrm{kg}.$$

The rest-mass of a neutron,

$$m_n = 1.674929 \times 10^{-27}\ \mathrm{kg}.$$

The speed of light (exact value, used to define the standard meter),

$$c = 2.99792458 \times 10^{8}\ \mathrm{m\,s^{-1}} = 2.99792458 \times 10^{10}\ \mathrm{cm\,s^{-1}}\,.$$

[1] From E. G. Cohen and B. N. Taylor, The 1986 Adjustment of the Fundamental Physical Constants, CODATA Bulletin, Number 63, November 1986.

The acceleration due to gravity near the earth's surface (varies slightly with latitude. This value applies near the latitude of Washington, DC, USA, or Madrid, Spain),

$$g = 9.80 \text{ m s}^{-7}.$$

The gravitational constant,

$$\mathrm{G} = 6.673 \times 10^{-11} \text{ m}^3\text{s}^{-2} \text{ kg}^{-1}.$$

The permittivity of a vacuum,

$$\epsilon_0 = 8.8545187817 \times 10^{-12}\mathrm{C}^2\mathrm{N}^{-1}\mathrm{m}^{-2}.$$

The permeability of a vacuum (exact value, by definition),

$$\mu_0 = 4\pi \times 10^{-7} \text{ NA}^{-2}.$$

Some Conversion Factors

1 pound = 1lb = 0.4535924 kg
1 inch = 1 in = 0.0254 m (exact value by definition)
1 calorie = 1 cal = 4.184 J (exact value by definition)
1 electron volt = 1 eV = 1.60219×10^{-19} J
1 erg = 10^{-7} J (exact value by definition)
1 atm = 760 torr = 101, 325 N m^{-2} = 101, 325 pascal (Pa) (exact values by definition)
1 atomic mass unit = 1 u = 1.66054×10^{-27} kg
1 horsepower = 1 hp = 745.700 watt = 745.700 J s^{-1}

B

Some Mathematical Formulas and Identities

1. The arithmetic progression of the first order to n terms,

$$a + (a + d) + (a + 2d) + \cdots + [a + (n - 1)d] = na + \frac{1}{2}n(n - 1)d$$
$$= \frac{n}{2}(\text{1st term} + n\text{th term}).$$

2. The geometric progression to n terms,

$$a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1 - r^n)}{1 - r}.$$

3. The definition of the arithmetic mean of $a_1, a_2, \ldots, a_n$,

$$\frac{1}{n}(a_1 + a_2 + \cdots + a_n).$$

4. The definition of the geometric mean of $a_1, a_2, \ldots, a_n$,

$$\bar{a}_G = (a_1 a_2 \ldots a_n)^{1/n}.$$

5. The definition of the harmonic mean of $a_1, a_2, \ldots, a_n$: If $\bar{a}_{\mathrm{H}}$ is the harmonic mean, then

$$\frac{1}{\bar{a}_H} = \frac{1}{n}\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots + \frac{1}{a_n}\right).$$

6. If

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots + b_n x^n$$

for all values of x, then

$$a_0 = b_0, \qquad a_1 = b_1 \qquad a_2 = b_2, \cdots, a_n = b_n.$$

Trigonometric Identities

7. $\sin^2(x) + \cos^2(x) = 1.$

8. $\tan(x) = \dfrac{\sin(x)}{\cos(x)}.$

9. $\mathrm{ctn}(x) = \dfrac{1}{\tan(x)}.$

10. $\sec(x) = \dfrac{1}{\cos(x)}.$

11. $\csc(x) = \dfrac{1}{\sin(x)}.$

12. $\sec^2(x) - \tan^2(x) = 1.$

13. $\csc^2(x) - \mathrm{ctn}^2(x) = 1.$

14. $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y).$

15. $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y).$ 16.

16. $\sin(2x) = 2\sin(x)\cos(x).$

17. $\cos(2x) = \cos^2(x) - \sin^2(x) = 1 - 2\sin^2(x).$

18. $\tan(x+y) = \dfrac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)},$

19. $\tan(2x) = \dfrac{2\tan(x)}{1-\tan^2(x)}.$

20. $\sin(x) = \dfrac{1}{2i}(e^{ix} - e^{-ix}).$

21. $\cos(x) = \dfrac{1}{2}(e^{ix} + e^{-ix}).$

22. $\sin(x) = -\sin(-x).$

23. $\cos(x) = \cos(-x).$

24. $\tan(x) = -\tan(-x).$

25. $\sin(ix) = i\sinh(x).$

26. $\cos(ix) = \cosh(x).$

27. $\tan(ix) = i\tanh(x).$

28. $\sin(x \pm iy) = \sin(x)\cosh(y) \pm i\cos(x)\sinh(y).$

29. $\cos(x \pm iy) = \cos(x)\cosh(y) \mp i\sin(x)\sinh(y).$

30. $\cosh(x) = \frac{1}{2}(e^x + e^{-x}).$

31. $\sinh(x) = \frac{1}{2}(e^x - e^{-x}).$

32. $\tanh(x) = \dfrac{\sinh(x)}{\cosh(x)}$.

33. $\mathrm{sech}(x) = \dfrac{1}{\cosh(x)}$.

34. $\mathrm{csch}(x) = \dfrac{1}{\sinh(x)}$.

35. $\mathrm{ctnh}(x) = \dfrac{1}{\tanh(x)}$.

36. $\cosh^2(x) - \sinh^2(x) = 1$.

37. $\tanh^2(x) + \mathrm{sech}^2(x) = 1$.

38. $\mathrm{ctnh}^2(x) - \mathrm{scsh}^2(x) = 1$.

39. $\sinh(x) = -\sinh(-x)$.

40. $\cosh(x) = \cosh(-x)$.

41. $\tanh(-x) = -\tanh(-x)$.

42. Relations obeyed by any triangle with angle A opposite side a, angle B opposite side b, and angle C opposite side c:

a) $A + B + C = 180^{\circ} = \pi$ rad

b) $c^2 = a^2 + b^2 - 2ab\cos(C)$

c) $\dfrac{a}{\sin(A)} = \dfrac{b}{\sin(B)} = \dfrac{c}{\sin(C)}$.

C

Infinite Series

C.1 Series with Constant Terms

1. $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \infty$.

2. $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$.

3. $1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots = \frac{\pi^4}{90}$.

4. $1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots = \zeta(p)$.
The function $\zeta(p)$ is called the *Riemann zeta function*.[1]

5. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln(2)$.

6. $1 - \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots = (1 - \frac{2}{2^p})\zeta(p)$.

C.2 Power Series

7. Maclaurin's series. If there is a power series in x for $f(x)$, it is

$$f(x) = f(0) + \left.\frac{df}{dx}\right|_{x=0} x = \frac{1}{2!}\left.\frac{d^2 f}{dx^2}\right|_{x=0} x^2 = \frac{1}{3!}\left.\frac{d^3 f}{dx^3}\right|_{x=0} x^3 + \cdots .$$

8. Taylor's series. If there is a power series in $x - a$ for $f(x)$, it is

$$f(x) = f(a) + \left.\frac{df}{dx}\right|_{x=a} (x - a) + \frac{1}{2!}\left.\frac{d^2 f}{dx^2}\right|_{x=a} (x - a^2) + \cdots .$$

In Eqs. (7) and (8), $\left.\frac{df}{dx}\right|_{x=a}$ means the value of the derivative df/dx evaluated at $x = a$.

[1]See H. B. Dwight, *Tables of Elementary and Some Higher Mathematical Functions*, 2nd ed., Dover, New York, 1958, for tables of values of this function.

9. If, for all values of x,

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots = b_0 + b_1x + b_2x^2 + b_3x^3 + \cdots$$

then $a_0 = b_0$, $a_1 = b_1$, $a_2 = b_2$, etc.

10. The reversion of a series. If

$$y = ax + bx^2 + cx^3 + \cdots$$

and

$$x = Ay + By^2 + Cy^3 + \cdots ,$$

then

$$\begin{aligned} A &= \frac{1}{a}, \quad B = -\frac{b}{a^3}, \quad C = \frac{1}{a^5}(2b^2 - ac), \\ D &= \frac{1}{a^7}(5abc - a^2d - 5b^3), \quad \text{etc.} \end{aligned}$$

See Dwight, *Table of Integrals and Other Mathematical Data* (cited above), for more coefficients.

11. Powers of a series. If

$$S = a + bx + cx^2 + dx^3 + \cdots ,$$

then

$$\begin{aligned} S^2 &= a^2 + 2abx + (b^2 + 2ac)x^2 + 2(ad + bc)x^3 \\ &\quad + (c^2 + 2ae + 2bd)x^4 + 2(af + be + cd)x^5 + \cdots \\ S^{1/2} &= a^{1/2}\left[1 + \frac{b}{2a}x + (\frac{2}{2a} - \frac{b^2}{8a^2})x^2 + \cdots\right] \\ S^{-1} &= a^{-1}\left[1 - \frac{b}{a}x + (\frac{b^2}{a^2} - \frac{c}{a})x^2 + (\frac{2bc}{a^2} - \frac{d}{a} - \frac{b^3}{a^3})x^3 + \cdots\right]. \end{aligned}$$

12. $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$.

13. $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$.

14. $\sin(\theta + x) = \sin(\theta) + x\cos(\theta) - \dfrac{x^2}{2!}\sin(\theta) - \dfrac{x^3}{3!}\cos(\theta) + \cdots$.

15. $\cos(\theta + x) = \cos(\theta) - x\sin(\theta) - \dfrac{x^2}{2!}\cos(\theta) + \dfrac{x^3}{3!}\sin(\theta) + \cdots$.

16. $\sin^{-1}(x) = x + \dfrac{x^3}{2 \cdot 3}. + \dfrac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} + \dfrac{1 \cdot 3 \cdot 5x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots$, where $x^2 < 1$. The series gives the principal value, $-\pi/2 < \sin^{-1}(x) < \pi/2$.

17. $\cos^{-1}(x) = \dfrac{\pi}{2} - \left(x + \dfrac{x^3}{2 \cdot 3} + \dfrac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} + \cdots\right)$, where $x^2 < 1$. The series gives the principal value, $0 < \cos^{-1}(x) < \pi$.

18. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots (x^2 < \infty)$.

19. $a^x = e^{x \ln(a)} = 1 + x \ln(a) + \frac{(x \ln(a))^2}{2!} + \cdots$.

20. $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \cdots$ ($x^2 < 1$ and $x = 1$).

21. $\ln(1 - x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\right)$($x^2 < 1$ and $x = -1$).

22. $\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots (x^2 < \infty)$.

23. $\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots (x^2 < \infty)$.

D

A Short Table of Derivatives

In the following list, a, b, and c are constants, and e is the base of natural logarithms.

1. $\dfrac{d}{dx}(au) = a\dfrac{du}{dx}$.

2. $\dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$.

3. $\dfrac{d}{dx}(uvw) = uv\dfrac{dw}{dx} + uw\dfrac{dv}{dx} + vw\dfrac{du}{dx}$.

4. $\dfrac{d(x^n)}{dx} = nx^{n-1}$.

5. $\dfrac{d}{dx}(\dfrac{u}{v}) = \dfrac{1}{v}\dfrac{du}{dx} - \dfrac{u}{v^2}\dfrac{dv}{dx} = \dfrac{1}{v^2}\left(v\dfrac{du}{dx} - u\dfrac{dv}{dx}\right)$.

6. $\dfrac{d}{dx}f(u) = \dfrac{df}{du}\dfrac{du}{dx}$, where f is some differentiable function of u and u is some differentiable function of x (the *chain rule*).

7. $\dfrac{d^2}{dx^2}f(u) = \dfrac{df}{du}\dfrac{d^2u}{dx^2} + \dfrac{d^2f}{du^2}\left(\dfrac{du}{dx}\right)^2$.

8. $\dfrac{d}{dx}\sin(ax) = a\cos(ax)$.

9. $\dfrac{d}{dx}\cos(ax) = -a\sin(ax)$.

10. $\dfrac{d}{dx}\tan(ax) = a\sec^2(ax)$.

11. $\dfrac{d}{dx}\mathrm{ctn}(ax) = -a\csc^2(ax)$.

12. $\dfrac{d}{dx}\sec(ax) = a\sec(ax)\tan(ax)$.

13. $\dfrac{d}{dx}\csc(ax) = -a\csc(ax)\operatorname{ctn}(ax).$

14. $\dfrac{d}{dx}\sin^{-1}\left(\dfrac{x}{a}\right) = \dfrac{1}{\sqrt{a^2 - x^2}}$ *if* x/a is in the first or fourth quadrant

$= \dfrac{-1}{\sqrt{a^2 - x^2}}$ *if* x/a is in the second or third quadrant.

15. $\dfrac{d}{dx}\cos^{-1}\left(\dfrac{x}{a}\right) = \dfrac{-1}{\sqrt{a^2 - x^2}}$ *if* x/a is in the first or second quadrant

$= \dfrac{1}{\sqrt{a^2 - x^2}}$ *if* x/a is in the third or fourth quadrant.

16. $\dfrac{d}{dx}\tan^{-1}\left(\dfrac{x}{a}\right) = \dfrac{a}{a^2 + x^2}.$

17. $\dfrac{d}{dx}\operatorname{ctn}^{-1}\left(\dfrac{x}{a}\right) = \dfrac{-a}{a^2 + x^2}.$

18. $\dfrac{d}{dx}e^{ax} = ae^{ax}.$

19. $\dfrac{d}{dx}a^{x} = a^{x}\ln(a).$

20. $\dfrac{d}{dx}a^{cx} = ca^{cx}\ln(a).$

21. $\dfrac{d}{dx}u^{y} = yu^{y-1}\dfrac{d}{dx} + u^{y}\ln(u)\dfrac{d}{dx}.$

22. $\dfrac{d}{dx}x^{x} = x^{x}[1 + \ln(x)].$

23. $\dfrac{d}{dx}\ln(ax) = \dfrac{1}{x}.$

24. $\dfrac{d}{dx}\log_a(x) = \dfrac{\log_a(a)}{x}.$

25. $\dfrac{d}{dq}\displaystyle\int_p^q f(x)\,dx = f(q)$ if p is independent of q.

26. $\dfrac{d}{dq}\displaystyle\int_p^q f(x)\,dx = -f(p)$ if q is independent of p.

E

A Short Table of Indefinite Integrals

In the following, an arbitrary constant of integration is to be added to each equation. a, b, c, g, and n are constants.

1. $\int dx = x.$

2. $\int x\, dx = \frac{x^2}{2}.$

3. $\int \frac{1}{x}\, dx = \ln(|x|)$ Do not integrate from negative to positive values of x.

4. $\int x^n\, dx = \frac{x^{n+1}}{n+1}$, where $n \neq -1$.

5. $\int (a+bx)^n\, dx = \frac{(a+bx)^{n+1}}{b(n+1)}.$

6. $\int \frac{1}{(a+bx)}\, dx = \frac{1}{b}\ln(|a+bx|).$

7. $\int \frac{1}{(a+bx)^n}\, dx = \frac{-1}{(n-1)b(a+bx)^{n-1}}.$

8. $\int \frac{x}{(a+bx)}\, dx = \frac{1}{b^2}\left[(a+bx) - a\ln(|a+bx|)\right].$

9. $\int \frac{a+bx}{c+gx}\, dx = \frac{bx}{g} + \frac{ag-bc}{g^2}\ln(|c+gx|).$

10. $\int \frac{1}{(a+bx)(c+gx)}\, dx = \frac{1}{ag-bc}\ln\left(\left|\frac{c+gx}{cx+bx}\right|\right).$

11. $\int \frac{1}{a^2+x^2}\, dx = \frac{1}{a}\tan^{-1}(\frac{x}{a}).$

12. $\displaystyle\int \frac{x}{(a^2+x^2)^2}\,dx = \frac{-1}{2(a^2+x^2)}.$

13. $\displaystyle\int \frac{x}{(a^2+x^2)}\,dx = \frac{1}{2}\ln(a^2+x^2).$

14. $\displaystyle\int \frac{1}{(a^2-b^2x^2)}\,dx = \frac{1}{2ab}\ln\left(\left|\frac{a+bx}{a-bx}\right|\right).$

15. $\displaystyle\int \frac{x}{(a^2-x^2)}\,dx = -\frac{1}{2}\ln(|a^2-x^2|).$

16. $\displaystyle\int \frac{x^{1/2}}{(a^2+b^2x)}\,dx = \frac{2x^{1/2}}{b^2} - \frac{2a}{b^3}\tan^{-1}\left(\frac{bx^{1/2}}{a}\right).$

17. $\displaystyle\int \frac{1}{(a+bx^2)^{\rho/2}}\,dx = \frac{-2}{(p-2)b(a+bx)^{(\rho-2)/2}}.$

18. $\displaystyle\int \frac{1}{(x^2+a^2)^{1/2}}\,dx = \ln(x+(x^2+a^2)^{1/2}).$

19. $\displaystyle\int \frac{x}{(x^2+a^2)^{1/2}}\,dx = (x^2+a^2)^{1/2}.$

20. $\displaystyle\int \frac{1}{(x^2-a^2)^{1/2}}\,dx = \ln(x+(x^2-a^2)^{1/2}).$

21. $\displaystyle\int \frac{x}{(x^2-a^2)^{1/2}}\,dx = (x^2-a^2)^{1/2}.$

22. $\displaystyle\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax).$

23. $\displaystyle\int \sin(a+bx)\,dx = -\tfrac{1}{b}\cos(a+bx).$

24. $\displaystyle\int x\sin(x)\,dx = \sin(x) - x\cos(x).$

25. $\displaystyle\int x^2\sin(x)\,dx = 2x\sin(x) - (x^2-2)\cos(x).$

26. $\displaystyle\int \sin^2(x)\,dx = \frac{x}{2} - \frac{\sin(2x)}{4} = \frac{x}{2} - \frac{\sin(x)\cos(x)}{2}.$

27. $\displaystyle\int x\sin^2(x)\,dx = \frac{x^2}{4} - \frac{x\sin(2x)}{4} - \frac{\cos(2x)}{8}.$

28. $\displaystyle\int \frac{1}{1+\sin(x)}\,dx = -\tan\left(\frac{\pi}{4} - \frac{x}{2}\right).$

29. $\displaystyle\int \cos(ax)\,dx = \frac{1}{a}\sin(ax).$

30. $\displaystyle\int \cos(a+bx)\,dx = \frac{1}{b}\sin(a+bx).$

31. $\int x\cos(x)\,dx = \cos(x) + x\sin(x).$

32. $\int x^2\cos(x)\,dx = 2x\cos(x) + (x^2 - 2)\sin(x).$

33. $\int \cos^2(x)\,dx = \dfrac{x}{2} + \dfrac{\sin(2x)}{4} = \dfrac{x}{2} + \dfrac{\sin(x)\cos(x)}{2}.$

34. $\int x\cos^2(x)\,dx = \dfrac{x^2}{4} + \dfrac{x\sin(2x)}{4} + \dfrac{\cos(2x)}{8}.$

35. $\int \dfrac{1}{1+\cos(x)}\,dx = \tan\left(\dfrac{x}{2}\right).$

36. $\int \sin(x)\cos(x)\,dx = \dfrac{\sin^2(x)}{2}.$

37. $\int \sin^2(x)\cos^2(x)\,dx = \dfrac{1}{8}\left[x - \dfrac{\sin(4x)}{4}\right].$

38. $\int \sin^{-1}\left(\dfrac{x}{a}\right)\,dx = x\sin^{-1}\left(\dfrac{x}{a}\right) + (a^2 - x^2)^{1/2}.$

39. $\int [\sin^{-1}\left(\dfrac{x}{a}\right)]^2\,dx = x[\sin^{-1}\left(\dfrac{x}{a}\right)]^2 - 2x + 2(a^2 - x^2)^{1/2}\sin^{-1}\left(\dfrac{x}{a}\right).$

40. $\int \cos^{-1}\left(\dfrac{x}{a}\right)\,dx = x\cos^{-1}\left(\dfrac{x}{a}\right) - (a^2 - x^2)^{1/2}.$

41. $\int [\cos^{-1}\left(\dfrac{x}{a}\right)]^2\,dx = x[\cos^{-1}\left(\dfrac{x}{a}\right)]^2 - 2x - 2(a^2 - x^2)^{1/2}\cos^{-1}\left(\dfrac{x}{a}\right).$

42. $\int \tan^{-1}\left(\dfrac{x}{a}\right)\,dx = x\tan^{-1}\left(\dfrac{x}{a}\right) - \dfrac{a}{2}\ln(a^2 + x^2).$

43. $\int x\tan^{-1}\left(\dfrac{x}{a}\right)\,dx = \tfrac{1}{2}(x^2 + a^2)\tan^{-1}\left(\dfrac{x}{a}\right) - \dfrac{ax}{2}.$

44. $\int e^{ax}\,dx = \dfrac{1}{a}e^{ax}.$

45. $\int a^r\,dx = \dfrac{a^x}{\ln(a)}.$

46. $\int xe^{ax}\,dx = e^{ax}\left(\dfrac{x}{a} - \dfrac{1}{a^2}\right).$

47. $\int x^2e^{ax}\,dx = e^{ax}\left[\dfrac{x^2}{a} - \dfrac{2x}{a^2} + \dfrac{2}{a^3}\right].$

48. $\int e^{ax}\sin(x)\,dx = \dfrac{e^{ax}}{a^2+1}[a\sin(x) - \cos(x)].$

49. $\int e^{ax}\cos(x)\,dx = \dfrac{e^{ax}}{a^2+1}[a\sin(x) + \sin(x)].$

50. $\int e^{ax}\sin^2(x)\,dx = \frac{e^{ax}}{a^2+4}\left[a\sin^2(x) - 2\sin(x)\cos(x) + \frac{2}{a}\right].$

51. $\int \ln(ax)\,dx = x\ln(ax) - x.$

52. $\int x\ln(x)\,dx = \frac{x^2}{2}\ln(x) - \frac{x^2}{4}.$

53. $\int \frac{\ln(ax)}{x}\,dx = \frac{1}{2}[\ln(ax)]^2.$

54. $\int \frac{1}{x\ln(x)}\,dx = \ln(|\ln(x)|).$

55. $\int \tan(ax)\,dx = \frac{1}{a}\ln(|\sec(ax)|) = -\frac{1}{a}\ln(|\cos(ax)|).$

56. $\int \cot(ax)\,dx = \frac{1}{a}\ln(|\sin(ax)|).$

F

A Short Table of Definite Integrals

In the following list, a, b, m, n, p, and r are constants.

1. $$\int_0^\infty x^{n-1}e^{-x}\,dx = \int_0^1 \left[\ln\left(\frac{1}{x}\right)\right]^{-1} dx = \Gamma(n) \quad (n > 0).$$
   The function $\Gamma(n)$ is called the *gamma function*. It has the following properties: for any $n > 0$,
   $$\Gamma(n+1) = n\Gamma(n).$$
   for any integral value of $n > 0$,
   $$\Gamma(n) = (n-1)!$$
   for n not an integer,
   $$\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin(n\pi)}$$
   $$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

2. $$\int_0^\infty \frac{1}{1+x+x^2}\,dx = \frac{\pi}{3\sqrt{3}}.$$

3. $$\int_0^\infty \frac{x^{p-1}}{(1+x)^\rho}\,dx = \frac{\pi}{\sin(p\pi)} \quad (0 < p < 1).$$

4. $$\int_0^\infty \frac{x^{p-1}}{a+x}\,dx = \frac{\pi a^{p-1}}{\sin(p\pi)} \quad (0 < p < 1).$$

5. $$\int_0^\infty \frac{x^p}{(1+ax)^2}\,dx = \frac{p\pi}{a^{\rho+1}\sin(p\pi)}.$$

6. $$\int_0^\infty \frac{1}{1+x^p}\,dx = \frac{\pi}{p\sin(\pi/p)}.$$

7. $$\int_0^{\pi/2} \sin^2(mx)\,dx = \int_0^{\pi/2} \cos^2(mx)\,dx = \frac{\pi}{4} \quad (m = 1, 2, \ldots).$$

8. $\displaystyle\int_0^{\pi} \sin^2(mx)\,dx = \int_0^{\pi} \cos^2(mx)\,dx = \frac{\pi}{2}\ (m = 1, 2, \ldots).$

9. $\displaystyle\int_0^{\pi/2} \tan^p(x)\,dx = \int_0^{\pi/2} \operatorname{ctn}^p(x)\,dx = \frac{\pi}{2\cos(p\pi/2)}\ (p^2 < 1).$

10. $\displaystyle\int_0^{\pi/2} \frac{x}{\tan(x)}\,dx = \frac{\pi}{2}\ln(2).$

11. $\displaystyle\int_0^{\pi/2} \sin^p(x)\cos^p(x)\,dx = \frac{\Gamma((p+1)/2)\,\Gamma((q+1)/2)}{2\Gamma((p+q)/2+1)} (p+1 > 0, q+1 > 0).$

12. $\displaystyle\int_0^{\pi} \sin(mx)\sin(nx)\,dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{\pi}{2} & \text{if } m = n \end{cases}\ (m, n \text{ integers}).$

13. $\displaystyle\int_0^{\pi} \cos(mx)\cos(nx)\,dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{\pi}{2} & \text{if } m = n \end{cases}\ (m, n \text{ integers}).$

14. $\displaystyle\int_0^{\pi} \sin(mx)\sin(nx)\,dx$

$$= \begin{cases} 0 & \text{if } m = n \\ 0 & \text{if } m \neq n \text{ and } m+n \text{ is even} \\ \frac{2m}{m^2-n^2} & \text{if } m \neq n \text{ and } m+n \text{ is odd } (m, n \text{ integers}). \end{cases}$$

15. $\displaystyle\int_0^{\infty} \sin\left(\frac{\pi x^2}{2}\right)dx = \int_0^{\infty} \cos\left(\frac{\pi x^2}{2}\right)dx = 1/2.$

16. $\displaystyle\int_0^{\infty} \sin(x^p)\,dx = \Gamma\left(1 + \frac{1}{p}\right)\sin\left(\frac{\pi}{2p}\right) \quad (p > 1).$

17. $\displaystyle\int_0^{\infty} \cos(x^p)\,dx = \Gamma\left(1 + \frac{1}{p}\right)\cos\left(\frac{\pi}{2p}\right) \quad (p > 1).$

18. $\displaystyle\int_0^{\infty} \frac{\sin(mx)}{x}\,dx = \begin{cases} \frac{\pi}{2} & \text{if } m > 0 \\ 0 & \text{if } m = 0 \\ -\frac{\pi}{2} & \text{if } m < 0 \end{cases}.$

19. $\displaystyle\int_0^{\infty} \frac{\sin(mx)}{x^{\rho}}\,dx = \frac{\pi m^{p-1}}{2\sin(p\pi/2)\Gamma(p)} \quad (0 < p < 2,\ m > 0).$

20. $\displaystyle\int_0^{\infty} e^{-ax}\,dx = \frac{1}{a} \quad (a > 0).$

21. $\displaystyle\int_0^{\infty} xe^{-ax}\,dx = \frac{1}{a^2} \quad (a > 0).$

22. $\displaystyle\int_0^{\infty} x^2e^{-ax}\,dx = \frac{2}{a^3} \quad (a > 0).$

23. $\int_0^\infty x^{1/2} e^{-ax}\, dx = \frac{\sqrt{\pi}}{2a^{3/2}} \quad (a > 0).$

24. $\int_0^\infty e^{-r^2 x^2}\, dx = \frac{\sqrt{\pi}}{2r} \quad (r > 0).$

25. $\int_0^\infty x e^{-r^2 x^2}\, dx = \frac{1}{2r^2} \quad (r > 0).$

26. $\int_0^\infty x^2 e^{-r^2 x^2}\, dx = \frac{\sqrt{\pi}}{4r^3} \quad (r > 0).$

27. $\int_0^\infty r^{2n+1} e^{-r^2 x^2}\, dx = \frac{n!}{2r^{2n+2}} \quad (r > 0,\ n = 1, 2, \ldots).$

28. $\int_0^\infty x^{2n} e^{-r^2 x^2}\, dx = \frac{(1)(3)(5)\cdots(2n-1)}{2^{n+1} r^{2n+1}} \sqrt{\pi} \quad (r > 0,\ n = 1, 2, \ldots).$

29. $\int_0^\infty x^a e^{-(rx)^b}\, dx = \frac{1}{br^{a+1}} \Gamma\left(\frac{a+1}{b}\right) \quad (a + 1 > 0,\ r > 0,\ b > 0).$

30. $\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\, dx = \ln\left(\frac{b}{a}\right).$

31. $\int_0^\infty e^{-ax} \sin(mx)\, dx = \frac{m}{a^2 + m^2} \quad (a > 0).$

32. $\int_0^\infty x e^{-ax} \sin(mx)\, dx = \frac{2am}{(a^2 + m^2)^2} \quad (a > 0).$

33. $\int_0^\infty x^{p-1} e^{-ax} \sin(mx)\, dx = \frac{\Gamma(p) \sin(p\theta)}{(a^2 + m^2)^{p/2}} \quad (a > 0,\ p > 0,\ m > 0)$, where $\sin(\theta) = m/r$, $\cos(\theta) = a/r$, $r = (a^2 + m^2)^{1/2}$.

34. $\int_0^\infty e^{-ax} \cos(mx)\, dx = \frac{a}{a^2 + m^2} \quad (a > 0).$

35. $\int_0^\infty x e^{-ax} \cos(mx)\, dx = \frac{a^2 - m^2}{(a^2 + m^2)^2} \quad (a > 0).$

36. $\int_0^\infty x^{p-1} e^{-ax} \cos(mx)\, dx = \frac{\Gamma(p) \cos(p\theta)}{(a^2 + m^2)^{\rho/2}} \quad (a > 0,\ p > 0)$, where θ is the same as given in Eq. (33).

37. $\int_0^\infty \frac{e^{-ax}}{x} \sin(mx)\, dx = \tan^{-1}\left(\frac{m}{a}\right) \quad (a > 0).$

38. $\int_0^\infty \frac{e^{-ax}}{x} [\cos(mx) - \cos(nx)]\, dx = \frac{1}{2} \ln\left(\frac{a^2 + n^2}{a^2 + m^2}\right) \quad (a > 0).$

39. $\int_0^\infty e^{-ax} \cos^2(mx)\, dx = \frac{a^2 + 2m^2}{a(a^2 + 4m^2)} \quad (a > 0).$

40. $\int_0^\infty e^{-ax} \sin^2(mx)\, dx = \frac{2m^2}{a(a^2 + 4m^2)} \quad (a > 0).$

41. $\displaystyle\int_0^1 \left[\ln\left(\frac{1}{x}\right)\right]^q dx = \Gamma(q+1) \quad (q+1>0).$

42. $\displaystyle\int_0^1 x^p \ln\left(\frac{1}{x}\right) dx = \frac{1}{(p+1)^2} \quad (p+1>0).$

43. $\displaystyle\int_0^1 x^p \left[\ln\left(\frac{1}{x}\right)\right]^q dx = \frac{\Gamma(q+1)}{(p+1)^{q+1}} \quad (p+1>0,\ q+1>0).$

44. $\displaystyle\int_0^1 \ln(1-x)\, dx = -1.$

45. $\displaystyle\int_0^1 x\ln(1-x)\, dx = \frac{-3}{4}.$

46. $\displaystyle\int_0^1 \ln(1+x)\, dx = 2\ln(2) - 1.$

47. $\displaystyle\int_0^\infty e^{-ax^2} \cos(kx)\, dx = \frac{\sqrt{\pi}}{2\sqrt{a}} e^{-k^2/(4a)}.$

G

Some Integrals with Exponentials in the Integrands: The Error Function

We begin with the integral

$$\int_0^\infty e^{-x^2}\, dx = 1.$$

We compute the value of this integral by a trick, squaring the integral and changing variables:

$$I^2 = \left[\int_0^\infty e^{-x^2}\, dx\right]^2 = \int_0^\infty e^{-x^2}\, dx \int_0^\infty e^{-y^2} dy = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy.$$

We now change to polar coordinates,

$$\begin{aligned} I^2 &= \int_0^{\pi/2} \int_0^\infty e^{-\rho^2} \rho\, d\rho\, d\phi = \frac{\pi}{2} \int_0^\infty e^{-\rho^2} \rho\, d\rho \\ &= \frac{\pi}{2} \int_0^\infty \frac{1}{2} e^{-z}\, dz = \frac{\pi}{4}. \end{aligned}$$

Therefore,

$$I = \int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$$

and

$$\boxed{\int_0^\infty e^{-ax^2}\, dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}} \tag{A.1}$$

Another trick can be used to obtain the integral,

$$\int_0^\infty x^{2n} e^{-ax^2}\,dx,$$

where n is an integer. For $n = 1$,

$$\begin{aligned}\int_0^\infty x^2 e^{-ax^2}\,dx &= -\int_0^\infty \frac{d}{da}\left[e^{-ax^2}\right] dx = -\frac{d}{da}\int_0^\infty e^{-ax^2}\,dx \\ &= -\frac{d}{da}\left[\frac{1}{2}\sqrt{\frac{\pi}{a}}\right] = \frac{1}{4a}\sqrt{\frac{\pi}{a}} = \frac{\pi^{1/2}}{4a^{3/2}}.\end{aligned} \tag{A.2}$$

For n an integer greater than unity,

$$\int_0^\infty x^{2n} e^{-ax^2}\,dx = (-1)^n - \frac{d^n}{da^n}\left[\frac{1}{2}\sqrt{\frac{\pi}{a}}\right]. \tag{A.3}$$

Equations (A.2) and (A.3) depend on the interchange of the order of differentiation and integration. This can be done if an improper integral is uniformly convergent. The integral in Eq. (A.1) is uniformly convergent for all real values of a greater than zero. Similar integrals with odd powers of x are easier. By the method of substitution,

$$\int_0^\infty x e^{-ax^2}\,dx = \frac{1}{2a}\int_0^\infty e^{-y}\,dy = \frac{1}{2a}. \tag{A.4}$$

We can apply the trick of differentiating under the integral sign just as in Eq. (A.3) to obtain

$$\int_0^\infty x^{2n+1} e^{-ax^2}\,dx = (-1)^n \frac{d^n}{da^n}\left(\frac{1}{2a}\right). \tag{A.5}$$

The integrals with odd powers of x are related to the gamma function, defined in Appendix G. For example,

$$\int_0^\infty x^{2n+1} e^{-x^2}\,dx = \frac{1}{2}\int_0^\infty y^n e^{-y}\,dy = \frac{1}{2}\Gamma(n+1). \tag{A.6}$$

The Error Function

The indefinite integral

$$\int e^{-x^2}\,dx$$

has never been expressed as a closed form (a formula not involving an infinite series or something equivalent). The definite integral for limits other than 0 and ∞ is not obtainable in closed form. Because of the frequent occurrence of such definite integrals, tables of numerical approximations have been generated.[1] One form in which the tabulation is done is as the *error function*, denoted by $\operatorname{erf}(x)$ and defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2}\, dt.$$

As you can see from Eq. (A.1),

$$\lim_{x \to \infty} \operatorname{erf}(x) = 1.$$

The name "error function" is chosen because of its frequent use in probability calculations involving the Gaussian probability distribution. Another form giving the same information is the *normal probability integral*[2]

$$\frac{1}{\sqrt{2\pi}} \int_{-x}^{x} e^{-t^2/2}\, dt.$$

Values of the Error Function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt^*$$

x		0	1	2	3	4	5	6	7	8	9
0.0	0.0	000	113	226	338	451	564	676	789	901	*013
0.1	0.1	125	236	348	459	569	680	790	900	*009	*118
0.2	0.2	227	335	443	550	657	763	869	974	*079	*183
0.3	0.3	286	389	491	593	694	794	893	992	*090	*187
0.4	0.4	284	380	475	569	662	755	847	937	*027	*117
0.5	0.5	205	292	379	465	549	633	716	798	879	959
0.6	0.6	039	117	194	270	346	420	494	566	638	708
0.7		778	847	914	981	*047	*112	*175	*238	*300	*361
0.8	0.7	421	480	538	595	651	707	761	814	867	918
0.9		969	*019	*068	*116	*163	*209	*254	*299	*342	*385
1.0	0.8	427	468	508	548	586	624	661	698	733	768
1.1		802	835	868	900	931	961	991	*020	*048	*076
1.2	0.9	103	130	155	181	205	229	252	275	297	319
1.3		340	361	381	400	419	438	456	473	490	507
1.4	0.95	23	39	54	69	83	97	*11	*24	*37	*49
1.5	0.96	61	73	84	95	*06	*16	*26	*36	*45	*55
1.6	0.97	63	72	80	88	96	*04	*11	*18	*25	*32

Continued on next page

[1] Two commonly available sources are Eugene Jahnke and Fritz Emde, *Tables of Functions*. Dover, New York, 1945, and Milton Abramowitz and Irene A. Stegun, Eds., *Handbook of Mathematical Functions with Formulas, Graph and Mathematical Tables*, U.S. Government Printing Office, Washington, DC, 1964.

[2] See for example Herbert B. Dwight, *Tables of Integrals and Other Mathematical Data*, 4th ed., Macmillan, New York, 1961.

x		0	1	2	3	4	5	6	7	8	9
1.7	0.98	38	44	50	56	61	67	72	77	82	86
1.8		91	95	99	*03	*07	*11	*15	*18	*22	*25
1.9	0.99	28	31	34	37	39	42	44	47	49	51
2.0	0.995	32	52	72	91	*09	*26	*42	*58	*73	*88
2.1	0.997	02	15	28	41	53	64	75	85	95	*05
2.2	0.998	14	22	31	39	46	54	61	67	74	80
2.3		86	91	97	*02	*06	*11	*15	*20	*24	*28
2.4	0.999	31	35	38	41	44	47	50	52	55	57
2.5		59	61	63	65	67	69	71	72	74	75
2.6		76	78	79	80	81	82	83	84	85	86
2.7		87	87	88	89	89	90	91	91	92	92
2.8	0.9999	25	29	33	37	41	44	48	51	54	56
2.9		59	61	64	66	68	70	72	73	75	77

* From Eugene Jahnke and Fritz Emde, *Tables of Functions*, Dover Publications, New York, 1945, p. 24.

To use this table, obtain the first digits of erf(x) from column 2 and the remaining digits from the appropriate column. Entries marked with * correspond to the value in the next lower row of column 2.